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Equations for a vector–bispinor

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Abstract. The structure of possible equations for a vector–bispinor is examined. A systematic procedure is given for obtaining the covariant equations with a given particle and mass content. It is shown how to derive the particle content and masses for a given equation. The relation between the root method and the method based on spin-projection operators is given.

1. Introduction

The development of supergravity (van Nieuwenhuizen 1981) points out the importance of spin $\frac{3}{2}$ particles in particle physics. For that reason it is of great interest to investigate the possible descriptions of spin $\frac{3}{2}$ more thoroughly. From the different representations the most suitable for this purpose is the vector–bispinor representation, since it offers a minimal dimensional theory that can be derived from the Lagrangian.

The most familiar of the equations for a vector–bispinor is the Rarita–Schwinger equation (Rarita and Schwinger 1941), which describes the single spin $\frac{3}{2}$ state. The other equations for a vector–bispinor are multiparticle equations where one or two spin $\frac{1}{2}$ states are also present (Kôiv *et al* 1982a, b). Due to the acausality of the Rarita–Schwinger equation (Velo and Zwanziger 1969), it is important to investigate all the possible equations for a vector–bispinor. It turns out that in minimal electromagnetic coupling the acausality defects are present only in the Rarita–Schwinger case.

The algebraic structure of all possible equations for a vector–bispinor was established using the formalism of spin-projection operators in Kôiv *et al* (1982a, b). The mass spectrum of equations with Hermitian matrices was previously investigated by Biritz (1975c, 1979). The method based on spin-projection operators is very useful in pure algebraic investigations of equations. However, the covariant formalism based on Dirac matrices is more suitable for applications. In this paper we give the relation between these two formalisms; the prescription for a derivation and also a systematic study of equations for a vector–bispinor. The connection between the non-covariant matrix form and the covariant tensor/spinor form of some equations has been previously considered by Frank (1973) and Cox (1982). We also consider the root method which has been frequently used (Ogievetsky and Sokatchev 1977, Berends *et al* 1979, van Nieuwenhuizen 1981). It allows one to derive equations for different spins and is also applicable in the superfield case (Ogievetsky and Sokatchev 1977). However, the root method is not so universal as the ordinary method of spin-projection operators.

In spite of the fact that the multiparticle equations for a vector–bispinor are not used so often, it seems that they are also important. Firstly, the multiparticle equations

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are free from acausality. Secondly, the equations of supergravity are based on multiparticle equations. The massless spin $\frac{3}{2}$ gravitino equation of supergravity (van Nieuwenhuizen 1981) is based on the multiparticle equation (see § 6). The equations for the spinor superfield are reduced in the superspin 1 and 0 cases to the multiparticle equations (Loide and Suurvarik 1984).

A vector–bispinor is also exploited to describe spin $\frac{1}{2}$ particles only (Capri 1969, Chandrasekaran *et al* 1972, Loide and Loide 1977). A spin $\frac{1}{2}$ equation for a vector–bispinor is obtained in the superspin 0 case for the spinor superfield.

The paper is planned as follows. We begin with some necessary information concerning the formalism of spin-projection operators. Then the general description of equations for a vector–bispinor is given. Furthermore we analyse the root method and then put the theory into the covariant form. Following this, we give some examples of frequently used equations.

2. General formalism of spin projection operators

We shall begin with a brief discussion of the construction of wave equations, using the formalism of spin-projection operators. We start from the first-order equations

$$(\mathrm{i}\partial_\mu \beta^\mu - m)\psi = 0, \tag{2.1}$$

where ψ transforms according to some finite-dimensional representation of the Lorentz group. Matrices β^μ satisfy the following commutation relations

$$[S^{\mu\nu}, \beta^\rho] = \eta^{\nu\rho}\beta^\mu - \eta^{\mu\rho}\beta^\nu, \tag{2.2}$$

where $\eta^{\mu\nu} = \mathrm{diag}(+---)$. The generators of the Lorentz group $S^{\mu\nu}$ satisfy

$$[S^{\mu\nu}, S^{\rho\sigma}] = \eta^{\nu\rho}S^{\mu\sigma} + \eta^{\mu\sigma}S^{\nu\rho} - \eta^{\mu\rho}S^{\nu\sigma} - \eta^{\nu\sigma}S^{\mu\rho}. \tag{2.3}$$

When the ψ representation is fixed, the problem reduces to the derivation of β^0; the other matrices β^κ are then established from (1.2).

In the following we give the general prescription for deriving β^0 in the formalism of spin-projection operators (Loide 1972, Loide and Loide 1977, Biritz 1975a, b, c, 1979). The structure of β^0 is more transparent in the representation where ψ is decomposed into a direct sum of irreducible representations $i = (k_i, l_i)$: $1 \oplus 2 \oplus \ldots \oplus r$. Then β^0 is written as follows

$$\beta^0 = |a_{ij} t_{ij}|, \tag{2.4}$$

where a_{ij} are arbitrary free parameters. Matrices t_{ij} are expressed with the help of spin-projection operators t^s_{ij}

$$t_{ij} = \sum_s \alpha_{ij}(s) t^s_{ij}, \tag{2.5}$$

where the summation is over all common spins in representations i and j. Only those t_{ij} are non-zero which correspond to linked representations i and j.

The most relevant objects in our construction are the spin projection operators t^s_{ij} expressed with the help of Clebsch–Gordan coefficients

$$(t^s_{ij})_{c'd',a'b'} = \sum_\sigma \langle c'd'(i)|s\sigma\rangle\langle s\sigma|(j)a'b'\rangle. \tag{2.6}$$

Due to the properties of Clebsch–Gordan coefficients, t^s_{ij} satisfy

$$t^s_{ij}t^{s'}_{kl} = \delta_{jk}\delta_{ss'}t^s_{il}. \tag{2.7}$$

The coefficients $\alpha_{ij}(s)$ in (2.5) are not arbitrary, but are uniquely determined by i and j. In the case of the representation $i=(k, l)$, we have four different linked representations j: $(k+\frac{1}{2}, l+\frac{1}{2})$, $(k-\frac{1}{2}, l-\frac{1}{2})$, $(k-\frac{1}{2}, l+\frac{1}{2})$ and $(k+\frac{1}{2}, l-\frac{1}{2})$, and four different relations for $\alpha_{ij}(s)$, respectively,

$$\begin{aligned}
&(1) \quad \alpha_{ij}(s) = \left[\frac{(k+l+2+s)(k+l+1-s)}{2(k+l+1)}\right]^{1/2},\\
&(2) \quad \alpha_{ij}(s) = \left[\frac{(k+l+1+s)(k+l-s)}{2(k+l)}\right]^{1/2},\\
&(3) \quad \alpha_{ij}(s) = \left[\frac{(s+k-l)(s-k+l+1)}{2k(2l+1)}\right]^{1/2},\\
&(4) \quad \alpha_{ij}(s) = \left[\frac{(s-k+l)(s+k-l+1)}{(2k+1)2l}\right]^{1/2}.
\end{aligned} \tag{2.8}$$

It is useful to decompose β^0 into the sum

$$\beta^0 = \beta^{s_1} + \beta^{s_2} + \ldots + \beta^{s_k}, \tag{2.9}$$

where β^s includes spin-projection operators t^s_{ij} of a given spin s. Due to (2.7), we have $\beta^{s_i}\beta^{s_j} = \delta_{s_is_j}(\beta^{s_i})^2$ and the investigation of β^0 reduces to the investigation of matrices β^s. Moreover, due to (2.7) the investigation of β^s reduces to the investigation of some $n\times n$ matrix formed from the coefficients $a_{ij}\alpha_{ij}(s)$, n is the number of irreducible representations which carried spin s.

The eigenvalues of β^0 depend on the choice of free parameters a_{ij}. The mass spectrum of particles described by an equation (2.1) is determined by the non-zero eigenvalues of the β^0 matrix, $\pm\lambda$, as follows: $m_\lambda = m/\lambda$ (Corson 1953, Gel'fand *et al* 1963). Physical masses correspond to real λ.

Often the parity operator π and the hermitising operator Λ are needed. Then the ψ-representation is composed of mutually conjugate representations—with each representation (k, l) there is also the conjugated representation (l, k). The general form of π and Λ is the same, as for β^0

$$\pi = |d_{ij}\Lambda_{ij}|, \qquad \Lambda = |\rho_{ij}\Lambda_{ij}|, \tag{2.10}$$

where

$$\Lambda_{ij} = \sum_s \lambda_{ij}(s)t^s_{ij}. \tag{2.11}$$

$\Lambda_{ij}\neq 0$ only in the case of the following representations i and j: $i=(k, k)$, $j=(k, k)$ and $i=(k, l)$, $j=(l, k)$ $(k\neq l)$. The coefficients $\lambda_{ij}(s)$ must satisfy

$$\lambda_{ij}(s) = -\lambda_{ij}(s\pm 1). \tag{2.12}$$

If we demand invariance under the space reflections and derivability from the Lagrangian, we have the additional restrictions

$$[\beta^0, \pi] = 0, \qquad (\beta^0)^+\Lambda = \Lambda\beta^0. \tag{2.13}$$

The relations (2.13) restrict the choice of coefficients a_{ij}, d_{ij} and ρ_{ij}.

The equation (2.1) can be derived from the Lagrangian

$$L=\tfrac{1}{2}\mathrm{i}(\psi^{+}\Lambda\beta^{\mu}\partial_{\mu}\psi-\psi^{+}\Lambda\bar{\partial}_{\mu}\beta^{\mu}\psi)-m\psi^{+}\Lambda\psi. \tag{2.14}$$

The derivation of higher-order equations is in principle the same (Loide 1972). If we have the nth-order equation

$$(\mathrm{i}\partial_{\mu_1}\ldots\mathrm{i}\partial_{\mu_n}\beta^{\mu_1\ldots\mu_n}-m^{n})\psi=0, \tag{2.15}$$

the matrices $\beta^{\mu_1\ldots\mu_n}$ satisfy the following commutation relations

$$[S^{\mu\nu},\beta^{\mu_1\ldots\mu_n}]=\sum_{l}\eta^{\nu\mu_l}\beta^{\mu_1\ldots\mu\ldots\mu_n}-\eta^{\mu\mu_l}\beta^{\mu_1\ldots\nu\ldots\mu_n}. \tag{2.16}$$

The general form of $\beta^{0\ldots0}$ is the same as in the case of the first-order equation (2.1) and is given by (2.4); t_{ij} are expressed with the help of (2.5). The difference is that in the case of a given representation i there are more linked representations j, and the coefficients $\alpha_{ij}(s)$ are not in general uniquely determined. In the case of representations $i=(1,\frac{1}{2})$ and $j=(\frac{1}{2},1)$, for example, (2.8) shows that in the case of a first-order equation $t_{ij}=t_{ij}^{3/2}+\frac{1}{2}t_{ij}^{1/2}$, i.e. both spin-projection operators $t_{ij}^{3/2}$ and $t_{ij}^{1/2}$ are present. In the case of third-order equations there are no restrictions on $\alpha_{ij}(s)$ and one may set $t_{ij}=t_{ij}^{3/2}$ or $t_{ij}=t_{ij}^{1/2}$.

The question arises, why do we mention here the higher-order equations, since it is well known that the equations where $n>2$ give the mass spectrum where unphysical masses are also present. In the following we demonstrate that the root method operates in principle with matrices corresponding to higher-order equations. Also it should be mentioned that in papers where the formalism of spin-projection operators was firstly used (Weinberg 1964a, b, 1969, Pursey 1965, Tung 1966, 1967), the equations of special type, where $\beta^{0\ldots0}=\beta^{s}$, were considered (Loide 1972). One of the general relations should be separately mentioned: if we have two arbitrary irreducible representations $i=(k,l)$ and $j=(k',l')$, then for each t_{ij}^{s} we get the nth-order equation, where $n=2\min\{(k+k'),(l+l')\}$. In the case of the representations $i=(1,\frac{1}{2})$ and $j=(\frac{1}{2},1)$, for example, $n=3$, as we have mentioned above.

3. Equations for the vector–bispinor

In this section we shall illustrate how to exploit the above given formalism in the case of vector–bispinor $\psi_{\alpha\mu}$, and give the description of all possible equations with physical mass spectrum (Kôiv *et al* 1982a, b).

The vector–bispinor $\psi_{\alpha\mu}$ is decomposed as

$$[(\tfrac{1}{2},0)\oplus(0,\tfrac{1}{2})]\otimes(\tfrac{1}{2},\tfrac{1}{2})=(1,\tfrac{1}{2})\oplus(0,\tfrac{1}{2})\oplus(\tfrac{1}{2},0)\oplus(\tfrac{1}{2},1). \tag{3.1}$$

Denoting the representations $1=(1,\frac{1}{2})$, $2=(0,\frac{1}{2})$, $3=(\frac{1}{2},0)$, and $4=(\frac{1}{2},1)$ we may write $\beta^{0}=\beta^{3/2}+\beta^{1/2}$ in the following general form:

$$\beta^{3/2}=\begin{vmatrix}0&0&0&t_{14}^{3/2}\\0&0&0&0\\0&0&0&0\\t_{41}^{3/2}&0&0&0\end{vmatrix}, \tag{3.2}$$

$$\beta^{1/2}=\begin{vmatrix} 0 & 0 & at_{13}^{1/2} & \frac{1}{2}t_{14}^{1/2} \\ 0 & 0 & ct_{23}^{1/2} & bt_{24}^{1/2} \\ bt_{31}^{1/2} & ct_{32}^{1/2} & 0 & 0 \\ \frac{1}{2}t_{41}^{1/2} & at_{42}^{1/2} & 0 & 0 \end{vmatrix}. \tag{3.3}$$

The parity operator π, corresponding to our choice of free parameters a, b and c, is

$$\pi=\begin{vmatrix} 0 & 0 & 0 & t_{14}^{3/2}-t_{14}^{1/2} \\ 0 & 0 & -t_{23}^{1/2} & 0 \\ 0 & -t_{32}^{1/2} & 0 & 0 \\ t_{41}^{3/2}-t_{41}^{1/2} & 0 & 0 & 0 \end{vmatrix}. \tag{3.4}$$

The hermitising matrix is the following

$$\Lambda=\begin{vmatrix} 0 & 0 & 0 & \rho_1(t_{14}^{3/2}-t_{14}^{1/2}) \\ 0 & 0 & \rho_2 t_{23}^{1/2} & 0 \\ 0 & \rho_2 t_{32}^{1/2} & 0 & 0 \\ \rho_1(t_{41}^{3/2}-t_{41}^{1/2}) & 0 & 0 & 0 \end{vmatrix}. \tag{3.5}$$

Derivability from the Lagrangian imposes on the parameters the following conditions:

$$b^*\rho_2=-\rho_1 a, \qquad c \text{ real}. \tag{3.6}$$

As we have mentioned above, masses are determined by the non-zero eigenvalues of the β^0 matrix, $\pm\lambda$. In (3.2) we have chosen the non-zero eigenvalues of $\beta^{3/2}$ to be ± 1, which means that the mass of a spin $\frac{3}{2}$ particle is equal to m.

The investigation of non-zero eigenvalues of $\beta^{1/2}$ reduces to the investigation of a reduced matrix $\beta_{1/2}$ formed from the parameters in (3.3)

$$\beta_{1/2}=\begin{vmatrix} 0 & 0 & a & \frac{1}{2} \\ 0 & 0 & c & b \\ b & c & 0 & 0 \\ \frac{1}{2} & a & 0 & 0 \end{vmatrix}. \tag{3.7}$$

The characteristic polynomial of $\beta_{1/2}$ gives the following eigenvalues:

$$\begin{aligned} 2\lambda_{1,2}&=c+\tfrac{1}{2}\pm[(c-\tfrac{1}{2})^2+4ab]^{1/2}, \\ 2\lambda_{3,4}&=-(c+\tfrac{1}{2})\mp[(c-\tfrac{1}{2})^2+4ab]^{1/2}. \end{aligned} \tag{3.8}$$

As we can see the eigenvalues of $\beta_{1/2}$ depend on two real parameters ab and c. For physical mass values λ must be real, therefore the only restriction on ab and c is

$$4ab\geqslant -(c-\tfrac{1}{2})^2. \tag{3.9}$$

The physical values of parameters ab and c may be represented on an ab–c diagram (figure 1). From (3.9) the physical region is determined by the parabola

$$4ab=-(c-\tfrac{1}{2})^2. \tag{3.10}$$

The points on the parabola correspond to the coincident eigenvalues

$$\lambda'=\lambda''=|c+\tfrac{1}{2}| \tag{3.11}$$

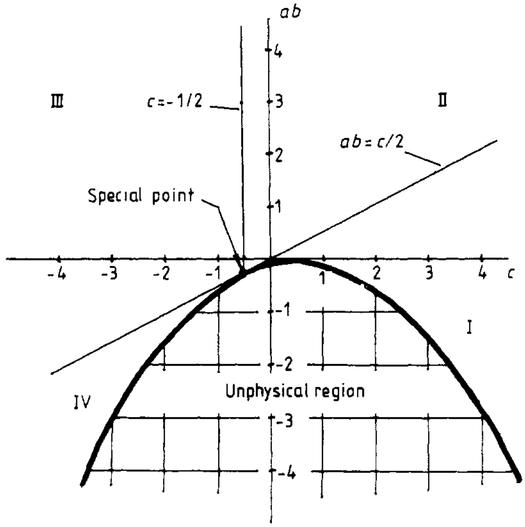

Figure 1. An ab–c diagram. On the line $c=-\frac{1}{2}$ there are points with the coincident eigenvalues $\lambda'=\lambda''$. On the line $ab=\frac{1}{2}c$ there are points where $\lambda'\neq 0$ and $\lambda''=0$. At the special point $\lambda'=\lambda''=0$.

(λ' and λ'' denote the non-negative eigenvalues from $\lambda_1,\ldots,\lambda_4$). These points must be also regarded as unphysical, since the minimal polynomial of $\beta_{1/2}$ is $(\beta_{1/2}^2-\lambda^2)^2=0$, and gives the vanishing charge and energy densities (Udgaonkar 1952, Cox 1977).

On the parabola (3.10) there is the special point

$$ab=-\tfrac{1}{4}, \qquad c=-\tfrac{1}{2}, \tag{3.12}$$

where $\lambda'=\lambda''=0$ and $\beta_{1/2}$ is nilpotent: $(\beta_{1/2})^2=0$. This special value of parameters is used when we want to describe a single spin $\frac{3}{2}$ particle. The equation we get is the well known Rarita–Schwinger spin $\frac{3}{2}$ equation. It is well known (Velo and Zwanziger 1969) that the Rarita–Schwinger field coupled minimally to an external electromagnetic field leads to acausality. When compared with the other physical equations for a vector–bispinor, it appears that all the other equations have no causality defects in minimal coupling since the β^0 are diagonalisable. All the equations with diagonalisable β-matrices are causal in the presence of an external electromagnetic field (Amar and Dozzio 1975). These causality considerations allow us to classify the equations with nilpotent matrices as unphysical (the other points on the parabola (3.10) are unphysical).

In the physical region there are two notable lines: $c=-\frac{1}{2}$ and $ab=\frac{1}{2}c$ (see figure 1). On the line $c=-\frac{1}{2}$ there are points with the coincident eigenvalues

$$\lambda'=\lambda''=(ab+\tfrac{1}{4})^{1/2}. \tag{3.13}$$

These points describe two spin $\frac{1}{2}$ particles with the same mass m/λ'.

On the line $ab=\frac{1}{2}c$ (which is a tangent of the parabola at the special point) there are points where

$$\lambda'=|c+\tfrac{1}{2}|, \qquad \lambda''=0. \tag{3.14}$$

These points describe one spin $\frac{1}{2}$ particle with mass m/λ'.

The remaining points in the physical region describe two spin $\frac{1}{2}$ particles with different masses m/λ' and m/λ''. While considering the parities, it is useful to distinguish four regions in the ab–c diagram:

region I	$c+\frac{1}{2}>0$,	$c>2ab$:	$\lambda_1^- > \lambda_2^- > 0$,
region II	$c+\frac{1}{2}>0$,	$c<2ab$:	$\lambda_1^- > \lambda_4^+ > 0$,
region III	$c+\frac{1}{2}<0$,	$c<2ab$:	$\lambda_4^+ > \lambda_1^- > 0$,
region IV	$c+\frac{1}{2}<0$,	$c>2ab$:	$\lambda_4^+ > \lambda_3^+ > 0$.

+ and − denote the parities of corresponding spin $\frac{1}{2}$ particles when the spin $\frac{3}{2}$ parity is chosen to be +1. Therefore, if $\lambda' \neq \lambda''$ we have four different parity combinations corresponding to the same mass spectrum.

From (3.8) it is possible to find the eigenvalues corresponding to the parameters ab and c. Sometimes one can choose the non-zero eigenvalues $\lambda' \neq \lambda''$; then the parameters are the following:

$$\begin{aligned}
&\text{region I} && ab=-\lambda_1\lambda_2+\tfrac{1}{2}(\lambda_1+\lambda_2)-\tfrac{1}{4}, && c=\lambda_1+\lambda_2-\tfrac{1}{2},\\
&\text{region II} && ab=\lambda_1\lambda_4+\tfrac{1}{2}(\lambda_1+\lambda_4)-\tfrac{1}{4}, && c=\lambda_1-\lambda_4-\tfrac{1}{2},\\
&\text{region III} && ab=\lambda_4\lambda_1-\tfrac{1}{2}(\lambda_4-\lambda_1)-\tfrac{1}{4}, && c=-(\lambda_4-\lambda_1)-\tfrac{1}{2},\\
&\text{region IV} && ab=-\lambda_4\lambda_3-\tfrac{1}{2}(\lambda_4+\lambda_3)-\tfrac{1}{4}, && c=-(\lambda_4+\lambda_3)-\tfrac{1}{2}.
\end{aligned} \tag{3.15}$$

It is also interesting to note that the points where one of the eigenvalues $\lambda>0$ is constant, lie on tangents of the parabola (3.10):

$$ab=(\lambda-\tfrac{1}{2})(\lambda-c), \qquad ab=(\lambda+\tfrac{1}{2})(\lambda+c). \tag{3.16}$$

The other eigenvalue varies from 0 to ∞. Therefore the masses of two spin $\frac{1}{2}$ particles may be chosen arbitrarily (the mass of a spin $\frac{3}{2}$ particle is equal to m).

As we have seen, one can construct equations with a different mass spectrum. In the case of the β^0 matrix (3.2) and (3.3), the equation describes one spin $\frac{3}{2}$ particle, and depending on the choice of free parameters ab and c two, one or no spin $\frac{1}{2}$ particles.

The vector–bispinor is also used to describe spin $\frac{1}{2}$ particles only (Capri 1969, Chandrasekaran *et al* 1972, Loide and Loide 1977). Then the representations $(1,\frac{1}{2})$ and $(\frac{1}{2},1)$ are not linked and therefore the β^0 matrix does not contain operators t_{14} and t_{41}, i.e. $a_{14}=a_{41}=0$. Now $\beta^0=\beta^{1/2}$, $\beta^{3/2}=0$, and the factor $\frac{1}{2}$ in (3.3) and (3.7) is absent. The physical region of parameters ab and c is determined by the parabola

$$4ab=-c^2. \tag{3.17}$$

The eigenvalues of the $\beta^{1/2}$ matrix are the following

$$2\lambda_{1,2}=c\pm(c^2+4ab)^{1/2}, \qquad 2\lambda_{3,4}=-c\mp(c^2+4ab)^{1/2}. \tag{3.18}$$

4. The root method

Recently the root method has been used (Ogievetsky and Sokatchev 1977, Berends *et al* 1979, van Nieuwenhuizen 1981). The root method allows us to derive equations for different spins, and is also applicable in the superfield case. Here we explain how the root method is connected with our previous formalism of spin-projection operators.

It turns out that this method is not so universal and seems somewhat artificial, since the derivation of equations in the ordinary way is quite simple.

The general idea of the root method (Ogievetsky and Sokatchev 1977) follows. Let Π^s be the projection operator which extracts spin s from the field ψ. Multiplying by $-\Box$ to a power q which is sufficient to cancel the non-locality of Π^s, we obtain the equation

$$(-\Box)^q \Pi^s \psi = (m^2)^q \psi. \tag{4.1}$$

The order of equation (4.1) is in general too high. Suppose $q = 1$ and a first-order equation is needed. Now the root method means that one must find an operator β defined by

$$\beta^2 = -\Box \Pi^s. \tag{4.2}$$

The first-order equation

$$\beta\psi = m\psi \tag{4.3}$$

reduces, using (4.2), to (4.1). The operator β defined by (4.2) is not in general unique.

In Berends *et al* (1979) this procedure is somewhat concretised. The field ψ transforms under some reducible representation and contains different spins. The most important objects we started with are the spin-projection operators Π^s_{ii} which extract spin s from some sub-representation when applied to ψ. Next, one must find the set of spin transition operators Π^s_{ij} $(i \neq j)$, so that operators Π^s_{ii} and Π^s_{ij} will satisfy

$$\Pi^s_{ij}\Pi^{s'}_{kl} = \delta_{ss'}\delta_{jk}\Pi^s_{il}. \tag{4.4}$$

Now, the root method means that it is necessary to find operators β^s_{ij} which satisfy

$$\begin{aligned} \Pi^s_{ij}\beta^{s'}_{kl} &= \delta_{ss'}\delta_{jk}\beta^s_{il}, \\ \beta^s_{ij}\Pi^{s'}_{kl} &= \delta_{ss'}\delta_{jk}\beta^s_{il}, \\ \beta^s_{ij}\beta^{s'}_{kl} &= \delta_{ss'}\delta_{jk}\Pi^s_{il}. \end{aligned} \tag{4.5}$$

Equation (4.3) is constructed, using β^s_{ij}, in the following way: β is a linear combination of operators β^s_{ij}

$$\beta = \sum_{ijs} a_{ij}(s)\beta^s_{ij}. \tag{4.6}$$

Using (4.5), one must calculate β^2 and choose the coefficients $a_{ij}(s)$ which lead to (4.2).

In the following we express the operators Π^s_{ij} and β^s_{ij} for a vector–bispinor field with the help of our previous spin-projection operators t^s. We restrict ourselves to the rest system ($\boldsymbol{p} = 0$). Since there is no unique prescription of how to find operators Π^s_{ij}, the most natural way to give Π^s_{ij} is the following

$$\Pi^{3/2}_{11} = \begin{vmatrix} t^{3/2}_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & t^{3/2}_{44} \end{vmatrix}, \qquad \Pi^{1/2}_{11} = \begin{vmatrix} t^{1/2}_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & t^{1/2}_{44} \end{vmatrix},$$

$$\Pi_{22}^{1/2}=\begin{vmatrix}0&0&0&0\\0&t_{22}^{1/2}&0&0\\0&0&t_{33}^{1/2}&0\\0&0&0&0\end{vmatrix},\qquad \Pi_{12}^{1/2}=\begin{vmatrix}0&t_{12}^{1/2}&0&0\\0&0&0&0\\0&0&0&0\\0&0&t_{43}^{1/2}&0\end{vmatrix},$$
$$\Pi_{21}^{1/2}=\begin{vmatrix}0&0&0&0\\t_{21}^{1/2}&0&0&0\\0&0&0&t_{34}^{1/2}\\0&0&0&0\end{vmatrix}. \tag{4.7}$$

From (4.7) we can see that $\Pi_{11}^{3/2}$ extracts spin $\frac{3}{2}$ and $\Pi_{11}^{1/2}$ extracts spin $\frac{1}{2}$ from the representation $(1,\frac{1}{2})\oplus(\frac{1}{2},1)$. Similarly, $\Pi_{22}^{1/2}$ extracts spin $\frac{1}{2}$ from bispinor $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$. Operators $\Pi_{12}^{1/2}$ and $\Pi_{21}^{1/2}$ relate linked representations $(1,\frac{1}{2})\oplus(\frac{1}{2},1)$ and $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$. It should be mentioned that the choice of operators Π_{ij}^{s} satisfying (4.4) is not unique and this non-uniqueness is connected with the spin $\frac{1}{2}$ degeneracy.

The operators β_{ij}^{s} which satisfy (4.5) are

$$\beta_{11}^{3/2}=\begin{vmatrix}0&0&0&t_{14}^{3/2}\\0&0&0&0\\0&0&0&0\\t_{41}^{3/2}&0&0&0\end{vmatrix},\qquad \beta_{11}^{1/2}=\begin{vmatrix}0&0&0&t_{14}^{1/2}\\0&0&0&0\\0&0&0&0\\t_{41}^{1/2}&0&0&0\end{vmatrix},$$
$$\beta_{22}^{1/2}=\begin{vmatrix}0&0&0&0\\0&0&t_{23}^{1/2}&0\\0&t_{32}^{1/2}&0&0\\0&0&0&0\end{vmatrix},\qquad \beta_{12}^{1/2}=\begin{vmatrix}0&0&t_{13}^{1/2}&0\\0&0&0&0\\0&0&0&0\\0&t_{42}^{1/2}&0&0\end{vmatrix},$$
$$\beta_{21}^{1/2}=\begin{vmatrix}0&0&0&0\\0&0&0&t_{24}^{1/2}\\t_{31}^{1/2}&0&0&0\\0&0&0&0\end{vmatrix}. \tag{4.8}$$

It is not, in general, easy to find β_{ij}^{s} if we start from operators Π_{ij}^{s}. In our case β_{ij}^{s} are related to Π_{ij}^{s} with the help of parity operator π: $\beta_{11}^{3/2}=\pi\Pi_{11}^{3/2}$, $\beta_{ij}^{1/2}=-\pi\Pi_{ij}^{1/2}$.

β^{s} matrices (3.2) and (3.3), π and Λ are expressed, using β_{ij}^{s}, in the following way:

$$\beta^{3/2}=\beta_{11}^{3/2},\qquad \beta^{1/2}=\tfrac{1}{2}\beta_{11}^{1/2}+c\beta_{22}^{1/2}+a\beta_{12}^{1/2}+b\beta_{21}^{1/2},$$
$$\pi=\beta_{11}^{3/2}-\beta_{11}^{1/2}-\beta_{22}^{1/2},\qquad \Lambda=\rho_1(\beta_{11}^{3/2}-\beta_{11}^{1/2})+\rho_2\beta_{22}^{1/2}. \tag{4.9}$$

The Rarita–Schwinger equation is now derived in the following way: we start from $\Pi_{11}^{3/2}$, and try to find $\beta^{0}=\beta^{3/2}+\beta^{1/2}$ whose square is equal to $\Pi_{11}^{3/2}$. Using (4.5) and (4.9), we obtain

$$(\beta^{0})^{2}=\Pi_{11}^{3/2}+(ab+\tfrac{1}{4})\Pi_{11}^{1/2}+(ab+c^{2})\Pi_{22}^{1/2}+a(c+\tfrac{1}{2})\Pi_{12}^{1/2}+b(c+\tfrac{1}{2})\Pi_{21}^{1/2}. \tag{4.10}$$

$(\beta^{0})^{2}=\Pi_{11}^{3/2}$ gives $ab=-\frac{1}{4}$ and $c=-\frac{1}{2}$. As we can see, the root method is indeed applicable. The trouble arises when we consider a multiparticle case. The problem is, how to find the combination of matrices Π_{ij}^{s}, the square root of which must be derived.

While making a comparison with the ordinary method described in § 2, we can see that it allows us to derive β^{0} directly, i.e. the operators β_{ij}^{s}. In the case of the root method one starts from operators Π_{ij}^{s} which are not needed in the construction of β^{0}, and the problem reduces to the derivation of the operators β^s_{ij}. Also, there is no general procedure for finding β^s_{ij}. In the covariant description, however, the root method is useful since one can more easily derive operators Π^s_{ij}, and then combine operators β^s_{ij} which satisfy (4.5).

5. Covariant γ-formalism

The method of spin-projection operators which we considered in § 2 is very useful in pure algebraic investigations of equations; the formalism of Dirac γ-matrices is easier to handle in the covariant description of equations for vector–bispinor $\psi_{\alpha\mu}$. In this section we transpose all the relations obtained earlier to the covariant form, and give the method for establishing the particle content and masses of an arbitrary first-order equation for the vector–bispinor.

We start from the operators Π^s_{ij}. The generators $S^{\mu\nu}$ of the Lorentz group are

$$(S^{\mu\nu})^{\varkappa}_{\lambda} = \eta^{\mu\varkappa}\eta^{\nu}_{\lambda} - \eta^{\nu\mu}\eta^{\mu}_{\lambda} + \tfrac{1}{2}\eta^{\varkappa}_{\lambda}\sigma^{\mu\nu}, \tag{5.1}$$

where $\sigma^{\mu\nu} = \frac{1}{2}[\gamma^{\mu}, \gamma^{\nu}]$, $\{\gamma^{\mu}, \gamma^{\nu}\} = 2\eta^{\mu\nu}$ and the spinor index is suppressed. Using the covariant spin operator S^2

$$S^2 = \tfrac{1}{2}(S^{\mu\nu}S_{\nu\mu} + (2\partial_{\mu}\partial^{\nu}/\Box)S^{\mu\rho}S_{\nu\rho}) \tag{5.2}$$

($\Box = \partial_{\mu}\partial^{\mu}$), which is decomposed as $S^2 = \frac{15}{4}\Pi^{3/2} + \frac{3}{4}\Pi^{1/2}$, we obtain the covariant spin-projection operators $\Pi^{3/2}$ and $\Pi^{1/2}$

$$(\Pi^{3/2})^{\varkappa}_{\lambda} = \eta^{\varkappa}_{\lambda} - \tfrac{1}{3}\gamma^{\varkappa}\gamma_{\lambda} - (2/3\Box)\partial^{\varkappa}\partial_{\lambda} + (\not\partial/3\Box)(\partial^{\varkappa}\gamma_{\lambda} - \partial_{\lambda}\gamma^{\varkappa}), \tag{5.3}$$

$$(\Pi^{1/2})^{\varkappa}_{\lambda} = \tfrac{1}{3}\gamma^{\varkappa}\gamma_{\lambda} + (2/3\Box)\partial^{\varkappa}\partial_{\lambda} - (\not\partial/3\Box)(\partial^{\varkappa}\gamma_{\lambda} - \partial_{\lambda}\gamma^{\varkappa}). \tag{5.4}$$

Obviously $\Pi^{3/2}$ is the covariant form of the spin-projection operator $\Pi^{3/2}_{11}$ given by (4.1). We denote it as $\Pi^{3/2}_{11}(\partial)$. In order to find the operators corresponds to $\Pi^{1/2}_{11}$ and $\Pi^{1/2}_{22}$, we must split $\Pi^{1/2}$ into two parts, one of which corresponds to the representation $(1, \frac{1}{2})\oplus(\frac{1}{2}, 1)$ and the other to the bispinor $(\frac{1}{2}, 0)\oplus(0, \frac{1}{2})$. $\Pi^{1/2}_{22}$ is easily extracted if we use the fact that in the case of the bispinor only the covariant spin operator is $S^2 = \frac{3}{4}I$, therefore $\Pi^{1/2}_{22}$ is the operator which seaprates $(\frac{1}{2}, 0)\oplus(0, \frac{1}{2})$. The bispinor is separated with the help of the first Casimir operator of the homogeneous Lorentz group: $C_1 = \frac{1}{2}S^{\mu\nu}S_{\nu\mu}$. The bispinor corresponds to the eigenvalue $\frac{3}{2}$ of C_1. We obtain $(\Pi^{1/2}_{22}(\partial))^{\varkappa}_{\lambda} = \frac{1}{4}\gamma^{\varkappa}\gamma_{\lambda}$; $\Pi^{1/2}_{12}$ and $\Pi^{1/2}_{21}$ are calculated from (4.3). The full set of covariant operators $\Pi^s_{ij}(\partial)$ corresponding to (4.7) is

$$\begin{aligned}
&(\Pi^{3/2}_{11}(\partial))^{\varkappa}_{\lambda} = \eta^{\varkappa}_{\lambda} - \tfrac{1}{3}\gamma^{\varkappa}\gamma_{\lambda} - (2/3\Box)\partial^{\varkappa}\partial_{\lambda}(\not\partial/3\Box)(\partial^{\varkappa}\gamma_{\lambda} - \partial_{\lambda}\gamma^{\varkappa}),\\
&(\Pi^{1/2}_{11}(\partial))^{\varkappa}_{\lambda} = \tfrac{1}{12}\gamma^{\varkappa}\gamma_{\lambda} + (2/3\Box)\partial^{\varkappa}\partial_{\lambda} - (\not\partial/3\Box)(\partial^{\varkappa}\gamma_{\lambda} - \partial_{\lambda}\gamma^{\varkappa}),\\
&(\Pi^{1/2}_{22}(\partial))^{\varkappa}_{\lambda} = \tfrac{1}{4}\gamma^{\varkappa}\gamma_{\lambda},\\
&(\Pi^{1/2}_{12}(\partial))^{\varkappa}_{\lambda} = -(1/4\sqrt{3})\gamma^{\varkappa}\gamma_{\lambda} + (1/\sqrt{3}\Box)\partial^{\varkappa}\not\partial\gamma_{\lambda},\\
&(\Pi^{1/2}_{21}(\partial))^{\varkappa}_{\lambda} = -(1/4\sqrt{3})\gamma^{\varkappa}\gamma_{\lambda} + (1/\sqrt{3}\Box)\gamma^{\varkappa}\not\partial\partial_{\lambda}.
\end{aligned} \tag{5.5}$$

Operators $\Pi^s_{ij}(\partial)$ are non-local, except for $\Pi^{1/2}_{22}(\partial)$, and contain $\Box^{-1}$.

It is not easy to find the square roots, since the non-local structure of β^s_{ij} is different. More easily, one can find the square root of $\Pi^{1/2}_{22}(\partial)$, which is equal to

$(\beta_{22}^{1/2}(\partial))^{\varkappa}_{\lambda}=\gamma^{\varkappa}\not\partial\gamma_{\lambda}/4\sqrt{\Box}$. The other $\beta_{ij}^{1/2}$ are calculated from (4.5). The full set of operators $\beta_{ij}^{s}(\partial)$ corresponding to (4.8) is

$$
\begin{aligned}
&(\beta_{11}^{3/2}(\partial))^{\varkappa}_{\lambda}=(1/\sqrt{\Box})(\not\partial\eta^{\varkappa}_{\lambda}-\tfrac{1}{3}\partial^{\varkappa}\gamma_{\lambda}-\tfrac{1}{3}\partial_{\lambda}\gamma^{\varkappa}+\tfrac{1}{3}\gamma^{\varkappa}\not\partial\gamma_{\lambda})-(2/3\Box^{3/2})\not\partial\partial^{\varkappa}\partial_{\lambda},\\
&(\beta_{11}^{1/2}(\partial))^{\varkappa}_{\lambda}=(1/3\sqrt{\Box})(\tfrac{1}{4}\gamma^{\varkappa}\not\partial\gamma_{\lambda}-\partial^{\varkappa}\gamma_{\lambda}-\partial_{\lambda}\gamma^{\varkappa})+(4/3\Box^{3/2})\not\partial\partial^{\varkappa}\partial_{\lambda},\\
&(\beta_{22}^{1/2}(\partial))^{\varkappa}_{\lambda}=(1/4\sqrt{\Box})\gamma^{\varkappa}\not\partial\gamma_{\lambda},\\
&(\beta_{12}^{1/2}(\partial))^{\varkappa}_{\lambda}=(1/\sqrt{3\Box})(\partial^{\varkappa}\gamma_{\lambda}-\tfrac{1}{4}\gamma^{\varkappa}\not\partial\gamma_{\lambda}),\\
&(\beta_{21}^{1/2}(\partial))^{\varkappa}_{\lambda}=(1/\sqrt{3\Box})(\gamma^{\varkappa}\partial_{\lambda}-\tfrac{1}{4}\gamma^{\mu}\not\partial\gamma_{\lambda}).
\end{aligned}
\tag{5.6}
$$

The non-local terms are $\Box^{-1/2}$ and $\Box^{-3/2}$. As we have mentioned in § 2, the spin-projection operators $t_{14}^{3/2}$, $t_{41}^{3/2}$, $t_{14}^{1/2}$ and $t_{41}^{1/2}$ (or $\beta_{11}^{3/2}$ and $\beta_{11}^{1/2}$) lead to a third-order equation because the non-local term $\Box^{-3/2}$ is present in operators $\beta_{11}^{3/2}(\partial)$ and $\beta_{11}^{1/2}(\partial)$. In order to obtain the first-order equation, one must eliminate these non-local terms. It is easy to see that the only possible combination is $\beta_{11}^{3/2}(\partial)+\frac{1}{2}\beta_{11}^{1/2}(\partial)$

$$
(\beta_{11}^{3/2}(\partial)+\tfrac{1}{2}\beta_{11}^{1/2}(\partial))^{\varkappa}_{\lambda}=(1/\sqrt{\Box})(\not\partial\eta^{\varkappa}_{\lambda}-\tfrac{1}{2}\gamma^{\varkappa}\partial_{\lambda}-\tfrac{1}{2}\partial^{\varkappa}\gamma_{\lambda}+\tfrac{3}{8}\gamma^{\varkappa}\not\partial\gamma_{\lambda}).
\tag{5.7}
$$

The reason why the spin-projection operators $t_{14}^{3/2}$ and $t_{14}^{1/2}$ are uniquely related in the first-order equation: $t_{14}=t_{14}^{3/2}+\frac{1}{2}t_{14}^{1/2}$, has now been explained.

The general first-order equation for $\psi_{\alpha\mu}$ is, from (4.9), (5.6) and (5.7), written in the following general form

$$
\mathrm{i}\sqrt{\Box}\,(\beta_{11}^{3/2}(\partial)+\tfrac{1}{2}\beta_{11}^{1/2}(\partial)+c\beta_{22}^{1/2}(\partial)+a\beta_{12}^{1/2}(\partial)+b\beta_{21}^{1/2}(\partial))^{\varkappa}_{\lambda}\psi^{\lambda}=m\psi^{\varkappa}.
\tag{5.8}
$$

Choosing the parameters a, b and c from the physical region (see figure 1), we can write down all the possible equations for $\psi_{\alpha\mu}$.

Usually the most general equation for $\psi_{\alpha\mu}$ is written as follows:

$$
(\mathrm{i}\not\partial-m)\psi^{\varkappa}+A\mathrm{i}\partial^{\varkappa}\gamma_{\lambda}\psi^{\lambda}+B\gamma^{\varkappa}\mathrm{i}\partial_{\lambda}\psi^{\lambda}+C\gamma^{\varkappa}(\mathrm{i}\not\partial)\gamma_{\lambda}\psi^{\lambda}=0
\tag{5.9}
$$

or in the standard form (2.1)

$$
\mathrm{i}(\partial_{\mu}\beta^{\mu})^{\varkappa}_{\lambda}\psi^{\lambda}=m\psi^{\varkappa},
$$

where

$$
(\beta^{\mu})^{\varkappa}_{\lambda}=\gamma^{\mu}\eta^{\varkappa}_{\lambda}+A\eta^{\mu\varkappa}\gamma_{\lambda}+B\gamma^{\varkappa}\eta^{\mu}_{\lambda}+C\gamma^{\varkappa}\gamma^{\mu}\gamma_{\lambda}.
\tag{5.10}
$$

It is easy to find a relation between the coefficients A, B, C and the parameters a, b, c:

$$
\begin{aligned}
&A=\tfrac{1}{6}(2\sqrt{3}\,a-3), && a=(6A+3)/2\sqrt{3},\\
&B=\tfrac{1}{6}(2\sqrt{3}\,b-3), && b=(6B+3)/2\sqrt{3},\\
&C=\tfrac{1}{24}[9+6c-2\sqrt{3}\,(a+b)], && c=\tfrac{1}{2}(2A+2B+8C-1).
\end{aligned}
\tag{5.11}
$$

The role of (5.11) is twofold—if we choose, using the formulae of § 3, the parameters a, b and c corresponding to a given mass spectrum, we can write down the general equation (5.9); on the other hand, if the equation (5.9) is given, we can find the coefficients a, b and c from A, B and C and therefore the particle content and mass spectrum of the given equation.

In order to write down π and Λ matrices, we first give the matrices β^s_{ij} in (4.8)

$$
\begin{aligned}
(\beta^{3/2}_{11})^{\varkappa}_{\lambda} &= -\tfrac{2}{3}\gamma^0\eta^{0\varkappa}\eta^0_{\lambda} + \gamma^0\eta^{\varkappa}_{\lambda} + \tfrac{1}{3}\gamma^{\varkappa}\gamma^0\gamma_{\lambda} - \tfrac{1}{3}\eta^{0\varkappa}\gamma_{\lambda} - \tfrac{1}{3}\gamma^{\varkappa}\eta^0_{\lambda},\\
(\beta^{1/2}_{11})^{\varkappa}_{\lambda} &= \tfrac{4}{3}\gamma^0\eta^{0\varkappa}\eta^0_{\lambda} - \tfrac{1}{3}\eta^{0\varkappa}\gamma_{\lambda} - \tfrac{1}{3}\gamma^{\varkappa}\eta^0_{\lambda} + \tfrac{1}{12}\gamma^{\varkappa}\gamma^0\gamma_{\lambda},\\
(\beta^{1/2}_{22})^{\varkappa}_{\lambda} &= \tfrac{1}{4}\gamma^{\varkappa}\gamma^0\gamma_{\lambda},\\
(\beta^{1/2}_{12})^{\varkappa}_{\lambda} &= (1/\sqrt{3})(\eta^{0\varkappa}\gamma_{\lambda} - \tfrac{1}{4}\gamma^{\varkappa}\gamma^0\gamma_{\lambda}),\\
(\beta^{1/2}_{21})^{\varkappa}_{\lambda} &= (1/\sqrt{3})(\gamma^{\varkappa}\eta^0_{\lambda} - \tfrac{1}{4}\gamma^{\varkappa}\gamma^0\gamma_{\lambda}).
\end{aligned}
\tag{5.12}
$$

The parity operator has a standard form. From (4.9) and (5.12) we obtain

$$(\pi)^{\varkappa}_{\lambda} = -2\gamma^0\eta^{0\varkappa}\eta^0_{\lambda} + \gamma^0\eta^{\varkappa}_{\lambda}. \tag{5.13}$$

The hermitising matrix Λ is given by (4.9). Due to (3.6), we obtain $\rho_2 = -(a/b^*)\rho_1$. We set $\rho_1 = 1$, and now from (4.9) and (5.12)

$$(\Lambda)^{\varkappa}_{\lambda} = -2\gamma^0\eta^{0\varkappa}\eta^0_{\lambda} + \gamma^0\eta^{\varkappa}_{\lambda} + \tfrac{1}{4}(1 - a/b^*)\gamma^{\varkappa}\gamma^0\gamma_{\lambda}. \tag{5.14}$$

In the case of $ab > 0$ the expression for Λ may be simplified. We may always operate with real a and b, and take $a = b$. Then $\Lambda = \pi$. If $ab < 0$, $\Lambda \neq \pi$.

While operating with equations and Lagrangians for the vector–bispinor $\psi_{\alpha\mu}$ we met with some ambiguity which we shall try to explain. Λ depends on the choice of β^0 matrix and satisfies $(\beta^0)^+\Lambda = \Lambda\beta^0$. On the other hand, it defines an invariant scalar product which is consistent with the Lagrangian (2.14). If we define the conjugated wavefunction

$$\tilde{\psi} = \psi^+\Lambda, \tag{5.15}$$

we have

$$\tilde{\psi}\psi \equiv \psi^+\Lambda\psi = -\bar{\psi}_{\mu}\psi^{\mu} + \tfrac{1}{4}(1 - a/b^*)\bar{\psi}_{\mu}\gamma^{\mu}\gamma_{\nu}\psi^{\nu} \tag{5.16}$$

where $\bar{\psi}_{\mu} = \psi^+_{\mu}\gamma^0$ is the Dirac conjugated wavefunction. Therefore, in general, $\tilde{\psi}\psi \neq -\bar{\psi}_{\mu}\psi^{\mu}$.

The Lagrangian (2.14) is written as

$$L = \tfrac{1}{2}\mathrm{i}(\tilde{\psi}\partial_{\mu}\beta^{\mu}\psi - \tilde{\psi}\bar{\partial}_{\mu}\beta^{\mu}\psi) - m\tilde{\psi}\psi. \tag{5.17}$$

Variation with respect to $\tilde{\psi}$ and ψ gives the equations

$$(\mathrm{i}\partial_{\mu}\beta^{\mu} - m)\psi = 0 \qquad \text{and} \qquad \tilde{\psi}(\mathrm{i}\bar{\partial}_{\mu}\beta^{\mu} + m) = 0. \tag{5.18}$$

Usually L is varied with respect to Dirac conjugated wavefunction $\bar{\psi}_{\mu}$, and then the equation obtained has a form

$$(\mathrm{i}\partial_{\mu}\alpha^{\mu} - mM)\psi = 0, \tag{5.19}$$

where $M \neq I$ is some non-singular matrix. The equation (5.18) is obtained if we multiply (5.19) from the left by M^{-1}.

As we have mentioned in § 3, $\psi_{\alpha\mu}$ is sometimes used to describe spin $\frac{1}{2}$ particles. These equations are written in the following general form:

$$A\mathrm{i}\partial^{\varkappa}\gamma_{\lambda}\psi^{\lambda} + B\gamma^{\varkappa}\mathrm{i}\partial_{\lambda}\psi^{\lambda} + C\gamma^{\varkappa}(\mathrm{i}\not{\partial})\gamma_{\lambda}\psi^{\lambda} = m\psi^{\varkappa}. \tag{5.20}$$

The parameters A, B, C and a, b, c are related in the following way:

$$a = \sqrt{3}\,A, \qquad b = \sqrt{3}\,B, \qquad c = A + B + 4C. \tag{5.21}$$

The eigenvalues of $\beta^{1/2}$ are calculated from (3.18).

Concluding this section, it is worth pointing out that the set of operators Π^s_{ij} and β^s_{ij} satisfying (4.4) and (4.5), is not unique. In § 4 we derived them, using the ideas based on the formalism of spin-projection operators t^s_{ij}. The operators $\Pi^s_{ij}(\partial)$ and $\beta^s_{ij}(\partial)$ in this section are derived similarly. Another set of operators P^s_{ij} is used by van Nieuwenhuizen (1981). These operators are related in the following way

$$\begin{aligned}
\Pi^{3/2}_{11} &= P^{3/2}_{11},\\
\Pi^{1/2}_{11} &= \tfrac{1}{4}P^{1/2}_{11} + \tfrac{3}{4}P^{1/2}_{22} - \tfrac{1}{4}\sqrt{3}(P^{1/2}_{12} + P^{1/2}_{21}),\\
\Pi^{1/2}_{22} &= \tfrac{3}{4}P^{1/2}_{11} + \tfrac{1}{4}P^{1/2}_{22} + \tfrac{1}{4}\sqrt{3}\,(P^{1/2}_{12} + P^{1/2}_{21}),\\
\Pi^{1/2}_{12} &= \tfrac{1}{4}\sqrt{3}\,(P^{1/2}_{22} - P^{1/2}_{11}) - \tfrac{1}{4}P^{1/2}_{12} + \tfrac{3}{4}P^{1/2}_{21},\\
\Pi^{1/2}_{21} &= \tfrac{1}{4}\sqrt{3}\,(P^{1/2}_{22} - P^{1/2}_{11}) + \tfrac{3}{4}P^{1/2}_{12} - \tfrac{1}{4}P^{1/2}_{21}.
\end{aligned} \tag{5.22}$$

6. Examples

In this section we give some examples of often used equations for the vector–bispinor $\psi_{\alpha\mu}$.

6.1. The Dirac equation

From a number of single mass equations there exists one which describes one spin $\frac{3}{2}$ and two spin $\frac{1}{2}$ particles with the same mass m (Loide and Loide 1977, Kôiv *et al* 1982a, b). This corresponds to the parameters $ab = \frac{3}{4}$ and $c = -\frac{1}{2}$. If we choose $a = b = \frac{1}{2}\sqrt{3}$, we obtain from (5.11) $A = B = C = 0$ and the equation (5.9) reduces to the Dirac equation

$$(\mathrm{i}\not\partial - m)\psi^\varkappa = 0. \tag{6.1}$$

If we add $\partial_\varkappa\psi^\varkappa = \gamma_\varkappa\psi^\varkappa = 0$, we obtain the Rarita–Schwinger equation (Rarita and Schwinger 1941).

6.2. The Rarita–Schwinger equation

The parameters corresponding to the Rarita–Schwinger equation are $ab = -\frac{1}{4}$, $c = -\frac{1}{2}$. The most natural choice is $a = -b = \frac{1}{2}$, but usually the non-symmetrical choice is used. The equation, given by Velo and Zwanziger (1969) corresponds to the parameters $a = -3/2\sqrt{3}$, $b = 1/2\sqrt{3}$ or $A = -1$, $B = -C = -\frac{1}{3}$, and the equation (5.9) reads

$$(\mathrm{i}\not\partial - m)\psi^\varkappa - \mathrm{i}\partial^\varkappa\gamma_\lambda\psi^\lambda - \tfrac{1}{3}\mathrm{i}\gamma^\varkappa\partial_\lambda\psi^\lambda + \tfrac{1}{3}\mathrm{i}\gamma^\varkappa\not\partial\gamma_\lambda\psi^\lambda = 0. \tag{6.2}$$

The equation given by Velo and Zwanziger (1969) is derived from the Lagrangian $L = \mathrm{i}\tilde{\psi}\partial_\mu\beta^\mu\psi - m\tilde{\psi}\psi$ but varied with respect to $\bar{\psi}_\mu$:

$$L = -\mathrm{i}\bar{\psi}_\mu\not\partial\psi^\mu + \mathrm{i}\bar{\psi}_\mu\partial^\mu\gamma_\lambda\psi^\lambda + \mathrm{i}\bar{\psi}_\mu\gamma^\mu\partial_\lambda\psi^\lambda - \mathrm{i}\bar{\psi}_\mu\gamma^\mu\not\partial\gamma_\lambda\psi^\lambda + m\bar{\psi}_\mu\psi^\mu - m\bar{\psi}_\mu\gamma^\mu\gamma_\lambda\psi^\lambda, \tag{6.3}$$

$$\mathrm{i}\not\partial\psi^\mu - \mathrm{i}\partial^\mu\gamma_\lambda\psi^\lambda - \mathrm{i}\gamma^\mu\partial_\lambda\psi^\lambda + \mathrm{i}\gamma^\mu\not\partial\gamma_\lambda\psi^\lambda - m(\eta^\mu_\lambda - \gamma^\mu\gamma_\lambda)\psi^\lambda = 0. \tag{6.4}$$

(6.4) is written in the form (5.19), where $M^\mu_\lambda = \eta^\mu_\lambda - \gamma^\mu\gamma_\lambda$. Multiplying (6.4) from the left by $(M^{-1})^\varkappa_\mu = \eta^\varkappa_\mu - \frac{1}{3}\gamma^\varkappa\gamma_\mu$, we obtain (6.2).

6.3. SO(1, 4)-type equation

In the papers by Kôiv *et al* (1970), Loide (1971) it is shown that the Rarita–Schwinger equation is equivalent to the following SO(1, 4)-type equation

$$(\mathrm{i}\not\partial - m)\psi^{\varkappa} - (1+\mathrm{i})\mathrm{i}\partial^{\varkappa}\gamma_{\lambda}\psi^{\lambda} - (1-\mathrm{i})\mathrm{i}\gamma^{\varkappa}\partial_{\lambda}\psi^{\lambda} + \gamma^{\varkappa}\not\partial\gamma_{\lambda}\psi^{\lambda} = 0, \tag{6.5}$$

with subsidiary conditions $\gamma_{\lambda}\psi^{\lambda} = \partial_{\lambda}\psi^{\lambda} = 0$. The equation (6.5) corresponds to the parameters $a = -\frac{1}{2}\sqrt{3}\,(1+2\mathrm{i})$, $b = -\frac{1}{2}\sqrt{3}\,(1-2\mathrm{i})$, $c = \frac{3}{2}$ and the matrices

$$S^{\mu 5} = \tfrac{1}{2}\beta^{\mu} \tag{6.6}$$

generate the SO(1, 4) algebra.

6.4.

The superfield equation for a spinor superfield, derived by Ogievetsky and Sokatchev (1977), leads in the case of superspin 1 to the following equation (Loide and Suurvarik 1984):

$$(\mathrm{i}\not\partial - m)\psi^{\varkappa} - \tfrac{1}{4}\mathrm{i}\gamma^{\varkappa}\partial_{\lambda}\psi^{\lambda} = 0. \tag{6.7}$$

Now we have $ab = \frac{3}{8}$, $c = -\frac{3}{4}$, which gives the masses m and $\frac{4}{3}m$. Equation (6.7), therefore, describes spin $\frac{3}{2}$ with mass m and two spins $\frac{1}{2}$ with masses m and $\frac{4}{3}m$. The state with mass $\frac{4}{3}m$ is eliminated by the additional condition $\partial_{\lambda}\psi^{\lambda} = 0$.

6.5.

In simple supergravity (van Nieuwenhuizen 1981), the massless gravitino field is described in the flat limit with the help of the following equation

$$\varepsilon^{\mu\nu\rho\sigma}\gamma^{5}\gamma_{\nu}\partial_{\rho}\psi_{\sigma} = 0. \tag{6.8}$$

It is interesting to note that the corresponding massive equation $\mathrm{i}\varepsilon^{\mu\nu\rho\sigma}\gamma^{5}\gamma_{\nu}\partial_{\rho}\psi_{\sigma} = m\psi^{\mu}$ which may be written as

$$(\mathrm{i}\not\partial - m)\psi^{\mu} - \mathrm{i}\partial^{\mu}\gamma_{\lambda}\psi^{\lambda} - \mathrm{i}\gamma^{\mu}\partial_{\lambda}\psi^{\lambda} + \mathrm{i}\gamma^{\mu}\not\partial\gamma_{\lambda}\psi^{\lambda} = 0 \tag{6.9}$$

gives us $ab = \frac{3}{4}$ and $c = \frac{3}{2}$. Now $ab = \frac{1}{2}c$, and therefore (6.9) describes one spin $\frac{3}{2}$ particle with mass m and one spin $\frac{1}{2}$ particle with mass $\frac{1}{2}m$.

6.6

From spin $\frac{1}{2}$ equations we note the one which corresponds to $a = b = 1$, $c = 0$ (Loide and Loide 1977). This equation describes two independent spin $\frac{1}{2}$ particles with the same mass m. In the representation where ψ is decomposed into a direct sum, we have two independent equations for representations $(1, \frac{1}{2}) \oplus (\frac{1}{2}, 0)$ and $(0, \frac{1}{2}) \oplus (\frac{1}{2}, 1)$. Using γ-matrices the corresponding equation reads

$$(\mathrm{i}/\sqrt{3})(\partial^{\varkappa}\gamma_{\lambda} + \gamma^{\varkappa}\partial_{\lambda} - \tfrac{1}{2}\gamma^{\varkappa}\not\partial\gamma_{\lambda})\psi^{\lambda} = m\psi^{\varkappa}. \tag{6.10}$$

6.7.

The superfield equation for superspin 0 (Loide and Suurvarik 1983) gives the following equation in the case of spinor superfield (Loide and Suurvarik 1984):

$$\tfrac{1}{3}\mathrm{i}(\partial^\varkappa - \gamma^\varkappa \not{\partial})\gamma_\lambda\psi^\lambda + \tfrac{1}{4}\mathrm{i}\gamma^\varkappa\partial_\lambda\psi^\lambda = m\psi^\varkappa. \tag{6.11}$$

Now $ab = \frac{1}{4}$, $c = -\frac{3}{4}$ and we have $\lambda' = 1$, $\lambda'' = \frac{1}{4}$ from (3.18). Equation (6.11) describes two spin $\frac{1}{2}$ particles with masses m and $4m$.

7. Conclusions

In this paper the full description of all equations for a vector–bispinor $\psi_{\alpha\mu}$ has been given. In order to simplify the use of the formulae given above we outline once more the general procedure for writing down an equation corresponding to a given mass spectrum, or to establish the particle content and masses of a given equation.

7.1. *Equations with given mass spectrum*

7.1.1. Spins $\frac{3}{2}$ and $\frac{1}{2}$. Spin $\frac{3}{2}$ has mass m; choose the masses of spin $\frac{1}{2}$ particles m/λ' and $m|\lambda''$. Parameters ab and c are determined from (3.13), (3.14) or (3.15). Now we choose a, b and c, coefficients A, B and C are calculated from (5.11), and the corresponding equation is given by (5.9).

The scalar product and Lagrangian are found with the help of (5.16) and (5.17), respectively.

7.1.2. Spin $\frac{1}{2}$. Choose the masses m/λ' and m/λ''. Parameters ab and c are determined from (3.18), A, B and C from (5.21). The corresponding equation is given by (5.20).

7.2. *Masses and particle content of a given equation*

7.2.1. Spins $\frac{3}{2}$ and $\frac{1}{2}$. Equation (5.9) is given; parameters a, b and c are then determined from (5.11). Formulae (3.8) give us the eigenvalues λ' and λ'', which in turn determine the masses and particle content.

7.2.2. Spin $\frac{1}{2}$. Equation (5.20) is given; parameters a, b and c are determined from (5.21) and eigenvalues λ' and λ'' from (3.18).

In conclusion we want to point out that the equations for a vector–bispinor in the special form (5.19) were previously analysed by Baisya (1971). This analysis corresponds to the following special choice of parameters: $A = -1$ ($a = -\frac{1}{2}\sqrt{3}$), and the other parameters are arbitrary. The specially mentioned cases II and III correspond to the lines $ab = \frac{1}{2}c$ and $c = -\frac{1}{2}$ in the ab–c diagram (see figure 1), respectively.

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